

# On general Plane Fronted Waves. Geodesics

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**Abstract.** A general class of Lorentzian metrics,  $\mathcal{M}_0 \times \mathbb{R}^2$ ,  $\langle \cdot, \cdot \rangle_z = \langle \cdot, \cdot \rangle_x + 2 du dv + H(x, u) du^2$ , with  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x)$  any Riemannian manifold, is introduced in order to generalize classical exact plane fronted waves. Here, we start a systematic study of their main geodesic properties: geodesic completeness, geodesic connectedness and multiplicity or causal character of connecting geodesics. These results are independent of the possibility of a full integration of geodesic equations. Variational and geometrical techniques are applied systematically. In particular, we prove that the asymptotic behavior of  $H(x, u)$  with  $x$  at infinity determines many properties of geodesics. Essentially, a subquadratic growth of  $H$  ensures geodesic completeness and connectedness, while the critical situation appears when  $H(x, u)$  behaves in some direction as  $|x|^2$ , as in the classical model of exact gravitational waves.

**Keywords:** Gravitational waves, plane fronted waves, geodesic connectedness, completeness, causal geodesics, variational methods, Ljusternik–Schnirelman theory.

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# 1 Introduction

Aim of this paper is to study some global properties of a family of Lorentzian manifolds which model plane fronted waves and, in particular, gravitational waves.

As explained in the classical book by Misner, Thorne and Wheeler [26], a gravitational wave represents ripples in the shape of spacetime which propagate across spacetime, as water waves are small ripples in the shape of the ocean's surface propagating across the ocean. The source of a gravitational wave is the motion of massive particles; in order to be detectable, very massive objects under violent dynamics must be involved (binary stars, supernovae, gravitational collapses of stars...). With more generality, plane fronted waves may also take into account the propagation of non-gravitational effects such as electromagnetism. *Exact plane fronted wave* solutions arise as a class of highly idealized standard models, as a compromise between reality and simplicity. Concretely, this model is a Lorentzian manifold  $(\mathbb{R}^4, ds^2)$  endowed with the metric

$$ds^2 = dx_1^2 + dx_2^2 + 2 du dv + H(x_1, x_2, u) du^2, \quad (1.1)$$

$(x_1, x_2, v, u) \in \mathbb{R}^4$ , where  $H : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a non-null smooth function. The scalar curvature of these spacetimes is always zero, but they are Ricci flat if and only if

$$\Delta_x H(x, u) \equiv 0, \quad (1.2)$$

where  $\Delta_x H$  denotes the Laplacian of  $H$  with respect to  $x = (x_1, x_2)$  (gravitational *pp*-wave). When one considers exact electromagnetic or gravitational waves,  $H(\cdot, u)$  is assumed to be a quadratic form on  $\mathbb{R}^2$ , with the additional assumption (1.2) in the gravitational case. Thus, an *exact (plane fronted) gravitational wave* is the particular spacetime obtained when the coefficient  $H$  in (1.1) has the special form

$$H(x_1, x_2, u) = f(u)(x_1^2 - x_2^2) + 2g(u) x_1 x_2 \quad (1.3)$$

for some  $f, g \in C^2(\mathbb{R}, \mathbb{R})$ ,  $f^2 + g^2 \not\equiv 0$  (an additional condition commonly used in this case is  $\int_{-\infty}^{+\infty} \sqrt{f^2(u) + g^2(u)} du < +\infty$ ). In particular, an exact gravitational wave is a *sandwich wave* if both  $f$  and  $g$  have compact support, while it is a *polarized wave* if  $g \equiv 0$ .

Historically, the study of gravitational waves goes back to Einstein (cf. [16]) but the standard exact model was already introduced by Brinkmann in order to determine Einstein spaces which can be improperly mapped conformally on some Einstein one (see [9]). Afterwards, they have been widely studied by many authors (see, for example, the summary in [35]). From the experimental viewpoint, the detection of gravitational waves is now one of the most exciting challenges<sup>1</sup>.

In a series of articles, Ehrlich and Emch studied systematically global properties of the exact model (cf. [13, 14, 15]). Especially, they studied geodesics of

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<sup>1</sup>Hulse and Taylor were awarded the Nobel Prize in 1993 for the seminal discovery in the seventies of indirect evidences of their existence; there is now a world wide effort -projects as LISA, GEO600, VIRGO, LIGO or TAMA300- for direct detection.

gravitational waves by means of the symmetries of the model, which allow an almost full integration of the geodesic equations. Summing up, important goals have been the characterization of properties such as causality, geodesic completeness, (non) geodesic connectedness or astigmatic conjugacy (all of them explained pedagogically in the book [4]).

Even though Ehrlich and Emch's study is very complete and exhaustive, the dependence of their results on the high degree of symmetry of the standard exact model must come in mind. Thus, in the present paper we study the following generalization of the classical model.

**Definition 1.1** *A semi-Riemannian manifold  $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$  is a plane fronted wave, briefly PFW, if there exists a connected  $n$ -dimensional Riemannian manifold  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x)$  such that it is  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  and*

$$\langle \cdot, \cdot \rangle_z = \langle \cdot, \cdot \rangle_x + 2 \, du \, dv + H(x, u) \, du^2, \quad (1.4)$$

where  $x \in \mathcal{M}_0$ , the variables  $(v, u)$  are the natural coordinates of  $\mathbb{R}^2$  and the smooth scalar field  $H : \mathcal{M}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  is such that  $H \not\equiv 0$ .

Let us remark that if  $\mathcal{M}_0 = \mathbb{R}^2$  is the standard 2-dimensional Euclidean space, the metric (1.4) reduces to (1.1) so, throughout all this paper, *exact PFW* will mean  $\mathbb{R}^4$  endowed with (1.1), while for an *exact gravitational wave* also (1.3) holds. Furthermore, for simplicity, in what follows the subscript of the metrics  $\langle \cdot, \cdot \rangle_z, \langle \cdot, \cdot \rangle_x$  ( $z = (x, v, u)$ ) will be dropped without possibility of confusion.

Definition 1.1 is a convenient generalization under both the physical and the mathematical viewpoint. Recall that, from the physical viewpoint, the existence of many symmetries must be regarded only as a first approach. But these symmetries cannot be expected to happen in an exact way; therefore, physical results must be independent of them in some reasonable sense. Our generalization retains the characteristic structure of an exact plane fronted wave, but it drops additional symmetries. Notice also that many authors have introduced modifications in the exact model in order to describe different situations such as, for example, colliding waves, or Schwarzschild or de Sitter background (see, e.g., [3, 7, 8, 11, 21, 23, 24, 35, 36]); moreover, the unrestricted topology of  $\mathcal{M}_0$  may be useful not only for such models but also for quantization (see, e.g., [17]). From the purely mathematical viewpoint, recall that now fundamental equations, as geodesic equations, cannot be integrated explicitly. Thus, one can see exactly the different mathematical tools needed for the different mathematical problems, as well as the exact relations among these results.

This paper and a forthcoming one (see [19]) are devoted to study systematically some geometrical properties of PFWs. In the present article, general properties and, especially, those ones concerning in geodesics are analyzed. It is organised as follows.

In Section 2, the Levi-Civita connection is determined. The scalar curvature is equal to that one of the Riemannian part and the assumptions for being Ricci flat or for satisfying the timelike convergence condition are given (see Propositions 2.1, 2.2 and Remark 2.3).

In Section 3, a preliminary study of geodesics is carried out. Geodesics in a PFW are related to trajectories of a particle on the Riemannian manifold

$(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  under a potential  $V_\Delta$  depending on time, where  $V_\Delta$  is essentially equal to  $-H$  (see Proposition 3.1). As a consequence, geodesic completeness is related to the completeness of trajectories for this potential (see Theorem 3.2). A natural condition for geodesic completeness is then *positive completeness* which holds, in particular, if  $H(x, u)$  does not increase superquadratically with  $x$  in some direction and its growth with respect to  $u$  is bounded (see Proposition 3.3, Corollary 3.4). Remarkably, for an exact gravitational wave the behavior of the function  $H$  with respect to the variable  $u$  may lie out of these hypotheses. Even though its completeness can be obtained in a straightforward way (see Proposition 3.5), we emphasize that such a result relies on the very especial form of  $H$ .

Section 4 is devoted to connectedness by geodesics. As a difference with previous references on exact PFWs, the impossibility to integrate geodesic equations force us to use results from global variational methods and Ljusternik-Schnirelman theory.

In Subsection 4.1 geodesics connecting two fixed points  $z_0, z_1 \in \mathcal{M}$  are related to the existence of critical points for a Lagrangian functional  $\mathcal{J}_\Delta$  depending only on the Riemannian part and we prove (see Theorem 4.3, Corollary 4.5): (i) the existence of at least one connecting geodesic from  $z_0$  to  $z_1$  (i.e., geodesic connectedness) whenever  $H$  does not become negative and decreasing with  $x$  quadratically or faster (i.e., essentially,  $H(x, u) \geq -R_0(u)|x|^{2-\epsilon}$  for some  $\epsilon > 0$ ) and (ii) under the previous assumptions, the existence of infinitely many space-like connecting geodesics, if the topology of  $\mathcal{M}_0$  is not homotopically trivial. Let us point out that exact gravitational waves are examples of PFWs with a quadratic growth of  $H$  in some directions, *which are not geodesically connected*. Nevertheless, even in this case, Theorem 4.3 gives an estimate of which points can be connected by a geodesic, which is shown to be optimal (see Example 4.6).

In Subsection 4.2 connectedness by causal geodesics is studied. Recall that a classical theorem by Avez and Seifert asserts: *in a globally hyperbolic spacetime, any pair of causally related points (i.e., a pair of points which can be joined by a causal curve) can be joined by a causal length-maximizing geodesic* (see [2, 33]). It is known that this conclusion does not hold for an exact gravitational wave. Nevertheless, in Theorem 4.7 we give an optimal estimate for the points where the conclusion holds, valid in general PFWs, while in Theorem 4.10 and Corollary 4.11 some multiplicity results are stated.

In Subsection 4.3 all the results in the previous two subsections are applied to the special case of exact gravitational waves, giving an accurate estimate about which points can be geodesically connected with others, the causal character of the connecting geodesics and its possible multiplicity (see Corollary 4.12, Proposition 4.13 and Remark 4.14).

It is worth pointing out that all these results on geodesics depend only on the qualitative behavior of  $H$  at infinity. Thus, they are independent of a property such as the focusing effect of null geodesics, a folk characteristic property of exact gravitational waves since Penrose's article [29]. In fact, it is easy to find examples of PFWs (Ricci flat –vacuum– or satisfying the timelike convergence condition) whose geodesics satisfy only some selected properties (completeness,

geodesic connectedness). Further discussions in [19] will show how, although focusing effect is also related to properties as (the lack of) global hyperbolicity in the exact case, it is rather independent in the generic non-exact case.

## 2 Christoffel symbols and curvature

From now on, manifolds are  $C^3$  while functions are  $C^2$  as a simplification, even though we need only  $C^1$  for many purposes. Furthermore, we will say that a tangent vector  $w$  is lightlike if  $\langle w, w \rangle = 0$  and  $w \neq 0$ ; while  $w$  is causal if it is either lightlike or timelike (0 is spacelike).

Now, let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  be a PFW equipped with the metric (1.4). We can fix a time orientation on it such that the lightlike vector field  $\partial_v$  is past directed; thus, the lightlike vector field  $\partial_u - \frac{1}{2}H\partial_v$  will be future directed. It is easy to check that  $\partial_v$  is also a parallel vector field and  $\partial_v = \nabla u$ , where  $u$  is the projection

$$(x, v, u) \in \mathcal{M}_0 \times \mathbb{R}^2 \mapsto u \in \mathbb{R}.$$

Thus, for any future directed causal curve  $z(s) = (x(s), v(s), u(s))$ , there results

$$\dot{u}(s) = \langle \dot{z}(s), \partial_v \rangle \geq 0, \quad (2.1)$$

and the inequality is strict if  $z(s)$  is timelike (the assumed background for this paper can be found in well-known books as [4, 22, 27]).

Fix some local coordinates  $x^1, \dots, x^n$  with respect to the Riemannian part  $\mathcal{M}_0$ , as well as  $(v, u) \in \mathbb{R}^2$ . A direct computation shows that the non-necessarily null Christoffel's symbols are

$$\begin{aligned} \Gamma_{ij}^k &= \Gamma_{ij}^{k(R)} \quad \text{for all } k, i, j \in \{1, \dots, n\}, \\ \Gamma_{uj}^v &= \Gamma_{ju}^v = \frac{1}{2} \frac{\partial H}{\partial x^j}(x, u) \quad \text{for all } j \in \{1, \dots, n\}, \\ \Gamma_{uu}^v &= \frac{1}{2} \frac{\partial H}{\partial u}(x, u), \\ \Gamma_{uu}^k &= -\frac{1}{2} \sum_{m=1}^n g_{(R)}^{km} \frac{\partial H}{\partial x^m}(x, u) \quad \text{for all } k \in \{1, \dots, n\}, \end{aligned}$$

where  $(g_{(R)}^{ij})_{ij}$  is the inverse of the matrix associated to the Riemannian metric on  $\mathcal{M}_0$  and  $\Gamma_{ij}^{k(R)}$  are its Christoffel's symbols if  $k, i, j \in \{1, \dots, n\}$ . Thus, the only non-null components of the Ricci curvature of the metric are

$$\begin{aligned} R_{ij} &= R_{ij}^{(R)} \quad \text{for all } i, j \in \{1, \dots, n\}, \\ R_{uu} &= -\frac{1}{2} \left( \sum_{k,l=1}^n \frac{\partial}{\partial x^k} (g_{(R)}^{kl} \frac{\partial H}{\partial x^l}(x, u)) + \sum_{k,a,l=1}^n g_{(R)}^{al} \frac{\partial H}{\partial x^l}(x, u) \Gamma_{ka}^{k(R)} \right) \\ &= -\frac{1}{2} \Delta_x H(x, u), \end{aligned}$$

where  $R_{ij}^{(R)}$  and  $\Delta_x$  are the components of the Ricci curvature and the Laplacian, respectively, associated to the Riemannian metric on  $\mathcal{M}_0$ . Thus, the (local form of the) Ricci curvature is

$$\text{Ric} = \sum_{i,j=1}^n R_{ij}^{(R)} dx^i \otimes dx^j - \frac{1}{2} \Delta_x H du \otimes du. \quad (2.2)$$

From (2.2) it is easy to check the following two propositions:

**Proposition 2.1** *In a PFW:*

- (i) *the scalar curvature at each  $(x, v, u)$  is equal to the scalar curvature of the Riemannian part  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  at  $x$ ;*
- (ii) *the Ricci tensor  $\text{Ric}$  is null if and only if the Riemannian Ricci tensor  $\text{Ric}^{(R)}$  is null and  $\Delta_x H \equiv 0$ .*

**Proposition 2.2** *A PFW satisfies the timelike convergence condition (i.e., for all timelike vector  $\zeta$ ,  $\text{Ric}(\zeta, \zeta) \geq 0$ ) if and only if for all  $(x, u) \in \mathcal{M}_0 \times \mathbb{R}$  and  $w \in T_x \mathcal{M}_0$ ,  $w \neq 0$ , it is*

$$\Delta_x H(x, u) \leq 0 \quad \text{and} \quad (2.3)$$

$$\text{Ric}^{(R)}(w, w) \geq 0. \quad (2.4)$$

**Remark 2.3** It is well-known that if a function  $f$  on a connected Riemannian manifold satisfies  $\Delta f \leq 0$  (or  $\geq 0$ ) and  $\nabla f \equiv 0$  out of a compact subset, then  $f$  is constant. Thus, if for each fixed  $u$ ,  $\nabla_x H(\cdot, u)$  is zero out of a compact subset (in particular, if  $\mathcal{M}_0$  is compact), then condition (2.3) implies  $H(x, u) \equiv H(u)$ .

If  $\text{Ric}^{(R)}(w, w) \geq \epsilon > 0$  for all unit  $w$  and the Riemannian metric on  $\mathcal{M}_0$  is complete, then  $\mathcal{M}_0$  is compact by Bonnet–Myers theorem; thus, condition (2.3) would imply that  $H$  is independent of  $x$  (notice that  $\text{Ric}^{(R)}(w, w) > 0$  for all  $w$  is possible in a complete non-compact manifold; for example, this happens in a paraboloid).

Recall that, when  $H(x, u) \equiv H(u)$ , the corresponding PFW is the product of the Riemannian part  $\mathcal{M}_0$  by  $\mathbb{R}^2$  endowed with the metric  $2dudv + H(u)du^2$  (such a bidimensional metric is flat, but simple natural extensions has its own interest, cf. [30, 32]).

### 3 General behavior of geodesics. Completeness

Aim of this section is analyzing the behavior of geodesics in a PFW and, in particular, studying their completeness, pointing out some sufficient conditions to geodesic completeness of the manifold.

**Proposition 3.1** *Let  $z : ]a, b[ \rightarrow \mathcal{M}$ ,  $z(s) = (x(s), v(s), u(s))$  ( $]a, b[ \subseteq \mathbb{R}$ ), be a curve on  $\mathcal{M}$  with constant energy  $\langle \dot{z}(s), \dot{z}(s) \rangle = E_z$  for all  $s \in ]a, b[$ . Assume  $0 \in ]a, b[$ . Then,  $z$  is a geodesic on  $\mathcal{M}$  if and only if the three following conditions hold:*

(a)  $u = u(s)$  is affine, i.e.,  $u(s) = u_0 + s\Delta u$  for all  $s \in ]a, b[$ , where  $u_0 = u(0)$ ,  $\Delta u = \dot{u}(0)$ ;

(b)  $x = x(s)$  is a solution of

$$D_s \dot{x} = -\nabla_x V_\Delta(x(s), s) \quad \text{for all } s \in ]a, b[, \quad (3.1)$$

where

$$V_\Delta(x, s) = -\frac{(\Delta u)^2}{2} H(x, u_0 + s\Delta u); \quad (3.2)$$

(c) if  $\Delta u = 0$  then  $v = v(s)$  is affine, i.e.,  $v(s) = v_0 + s\Delta v$  for all  $s \in ]a, b[$ , with  $v_0 = v(0)$ ,  $\Delta v = \dot{v}(0)$ ; otherwise, for all  $s \in ]a, b[$  it is

$$v(s) = v_0 + \frac{1}{2\Delta u} \int_0^s (E_z - \langle \dot{x}(\sigma), \dot{x}(\sigma) \rangle + 2V_\Delta(x(\sigma), \sigma)) d\sigma.$$

**Proof.** Fix local coordinates  $(x^1, \dots, x^n, v, u)$ , as in Section 2, and consider Christoffel's symbols of the metric (1.4); the geodesic equations become

$$\ddot{x}^i + \sum_{j,k=1}^n \Gamma_{jk}^{i(R)} \dot{x}^j \dot{x}^k + \Gamma_{uu}^i \dot{u}^2 = 0 \quad \text{for all } i \in \{1, \dots, n\}, \quad (3.3)$$

$$\ddot{v} + 2 \sum_{j=1}^n \Gamma_{ju}^v \dot{x}^j \dot{u} + \Gamma_{uu}^v \dot{u}^2 = 0, \quad (3.4)$$

$$\ddot{u} = 0. \quad (3.5)$$

Thus, (a) and (b) follows from (3.3), (3.5). For (c), the expression of the energy  $E_z$  yields

$$2\dot{v}\dot{u} = E_z - \langle \dot{x}, \dot{x} \rangle - H(x, u)\dot{u}^2. \quad (3.6)$$

So, if  $\Delta u = 0$  by (a) it is  $\dot{u} \equiv 0$ , so, use (3.4); otherwise, use (3.6). ■

Therefore, in order to investigate the properties of geodesics in a PFW, it is enough studying the behavior of the Riemannian trajectories under a suitable potential  $V \equiv V_\Delta$ . In particular, this happens for geodesic completeness, where we can take just  $V = -H$  (or, for convenience,  $V = -H/2$ ).

**Theorem 3.2** *A PFW is geodesically complete if and only if  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  is a complete Riemannian manifold and the trajectories of*

$$D_s \dot{x} = \frac{1}{2} \nabla_x H(x, s) \quad (3.7)$$

*are complete, i.e., each of them can be extended so as to be defined on all  $\mathbb{R}$ .*

**Proof.** The implication to the right is obvious by Proposition 3.1, as (3.7) is equivalent to (3.1) with  $u_0 = 0$  and  $\Delta u = 1$ , while each Riemannian geodesic  $x = x(s)$  in  $\mathcal{M}_0$  defines a trivial geodesic  $z = (x, 0, 0)$  in  $\mathcal{M}$ .

On the contrary, let  $x = x(s)$  be a solution of (3.1). Then, either  $\Delta u = 0$  and  $x$  is a Riemannian geodesic in the complete manifold  $\mathcal{M}_0$ , or  $\Delta u \neq 0$  and

$y(\sigma) = x((\sigma - u_0)/\Delta u)$  solves (3.7). In both these cases,  $x$  can be extended so as to be defined on all  $\mathbb{R}$ . ■

The completeness of the trajectories satisfying (3.7) has been studied by several authors (see, e.g., [12, 20, 34]). Frequently, they exploit the idea that the velocities of these trajectories are integral curves of a vector field  $X$  (the Lagrangian vector field) on the tangent manifold  $T\mathcal{M}_0$ . In general, if an integral curve  $c = c(s)$  of any vector field is defined on an interval  $[0, b[$ ,  $b < +\infty$ , and there exists a sequence  $s_n \rightarrow b$  such that  $\{c(s_n)\}_n$  converges, then  $c$  is extendible as an integral curve beyond  $b$  (symmetrically, if  $c$  is defined in  $]a, 0]$ ,  $a > -\infty$ ). Thus, one has just to ensure that the integral curves of  $X$  restricted to a bounded interval lie in a compact subset of  $T\mathcal{M}_0$ .

So, if, for example, the Riemannian part  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  is complete and the coefficient  $H(x, u)$  in (1.4) is autonomous (i.e., independent of  $u$ ), a natural condition for completeness of solutions of (3.7) is obtained assuming that  $H$  is controlled at infinity (in a suitable way) by a *positively complete* function  $U_0$ , i.e., a nonincreasing  $C^2$  function  $U_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$  ( $\mathbb{R}_+ = [0, +\infty[$ ) such that

$$\int_0^{+\infty} \frac{dt}{\sqrt{\alpha - U_0(t)}} = +\infty,$$

for some (and thus any)  $\alpha > U_0(0) = \sup U_0(\mathbb{R}_+)$ . More precisely, the following result can be stated (see [34] or also [1, Theorem 3.7.15]):

**Proposition 3.3** *Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  be a PFW such that  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  is complete and  $H$  is autonomous, i.e.,  $H(x, u) \equiv H(x)$ . If there exist  $r > 0$ ,  $\bar{x} \in \mathcal{M}_0$  and a positively complete function  $U_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that*

$$H(x) \leq -U_0(d(x, \bar{x})) \quad \text{for all } x \in \mathcal{M}_0 \text{ such that } d(x, \bar{x}) \geq r,$$

*then all the trajectories which satisfy (3.7) are complete.*

Let us point out that a simple example of positively complete function is  $U_0(t) = -R_0 t^p$  with  $R_0 > 0$  and  $0 \leq p \leq 2$ . So, Theorem 3.2 and Proposition 3.3 imply:

**Corollary 3.4** *Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  be a PFW such that  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  is complete and  $H(x, u) \equiv H(x)$ . If there exist  $r > 0$ ,  $\bar{x} \in \mathcal{M}_0$  and  $R_0 > 0$  such that*

$$H(x) \leq R_0 d^2(x, \bar{x}) \quad \text{for all } x \in \mathcal{M}_0 \text{ such that } d(x, \bar{x}) \geq r,$$

*then  $\mathcal{M}$  is geodesically complete.*

As pointed out in [20], the results on autonomous potentials imply also results on non-autonomous potentials by considering the product manifold  $\mathcal{M}_0 \times \mathbb{R}$ ; essentially, one has also to bound the growth of the potential  $V(x, u)$  with  $u$ . Nevertheless, the completeness of exact gravitational waves does not seem to be covered by these general results, at least if  $f$  and  $g$  in (1.3) are arbitrary (of course, the completeness would be straightforward for a sandwich wave). Anyway, it is easy to give particular results on geodesic completeness which cover all gravitational waves. But they rely in very particular expressions of



$H(x, u)$  and would not hold for an arbitrary (even exact) PFW. For the sake of completeness, we give such a result.

Consider  $\mathcal{M}_0 = \mathbb{R}^n$  with its classical Euclidean metric  $\langle \cdot, \cdot \rangle_0$ . Assume that the coefficient  $H(x, u)$  in (1.4) is in the canonical form for gravitational or electromagnetic waves, i.e., let

$$H(x, u) = \langle A(u)x, x \rangle_0, \quad (3.8)$$

where  $A(u)$  is a non-identically vanishing map from  $\mathbb{R}$  to  $M^{\text{sym}}(n, \mathbb{R})$ , the space of symmetric  $n \times n$  real valued matrices (if  $n = 2$  and  $\Delta_x H(x, u) \equiv 0$ , then we have exactly (1.3); compare with [13, Remark 2.3]). Under this assumption, it is  $\nabla_x H(x, u) = 2A(u)x$  for all  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ , so the equation (3.7) becomes

$$\ddot{x}(s) = A(s)x(s), \quad s \in \mathbb{R}. \quad (3.9)$$

A classical global existence theorem for linear ODEs implies that all the solutions of (3.9) are complete so the following result can be stated:

**Proposition 3.5** *Consider a PFW such that  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  is covered by Euclidean space  $\mathbb{R}^n$ , and  $H(x, u)$  is in the canonical form (3.8). Then, the PFW is geodesically complete.*

*In particular, any exact gravitational wave is geodesically complete.*

Finally, fix  $u_0 \in \mathbb{R}$  and consider the hypersurface

$$\Pi_{u_0} = \{(x, v, u) \in \mathcal{M} : u = u_0\}.$$

Clearly, the restriction of the metric to this hypersurface is degenerate positive-semidefinite, so it is not a semi-Riemannian submanifold. Nevertheless,  $\Pi_{u_0}$  is totally geodesic in the sense that taken any tangent vector  $w$  to  $\Pi_{u_0}$  the (necessarily non-timelike) geodesic in the PFW with initial velocity  $w$  remains in  $\Pi_{u_0}$ . More precisely, as a straightforward consequence of Proposition 3.1:

**Corollary 3.6** *Fix a vector  $w$  tangent to  $\mathcal{M}_0$ , and  $v_0, \Delta v \in \mathbb{R}$ . Let  $x(s)$  be the geodesic in  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  with initial velocity  $w$ . Then, the curve in  $\Pi_{u_0}$ ,  $z(s) = (x(s), v_0 + s\Delta v, u_0)$ , is a geodesic in the PFW.*

*Furthermore, a curve  $\gamma(s)$  in  $\Pi_{u_0}$  is lightlike (at any point) if and only if it is a reparametrization of a geodesic  $z(s) = (x(s), v_0 + s\Delta v, u_0)$  with constant  $x(s)$  and  $\Delta v \neq 0$ .*

## 4 Connection by geodesics

Geodesic connectedness of spacetimes has been widely studied under very different techniques (see, e.g., the survey [31]). In particular, since the seminal articles by Benci, Fortunato and Giannoni, variational methods have been extensively used in Lorentzian Geometry for this and other related problems (see [5, 6] or the book [25]). Nevertheless, here our viewpoint is rather different and relies exclusively in the previous results by the authors in [10], which are based in standard variational techniques and Ljusternik-Schnirelman theory as developed in references like [18, 28].

## 4.1 Geodesic connectedness

In what follows, put  $I = [0, 1]$  (closed interval) and recall that, in Proposition 3.1, if  $1 \in ]a, b[$  then  $\Delta u = u(1) - u(0)$ . As in the case of geodesic completeness, the problem of existence and multiplicity of connecting geodesics in a PFW reduces to the existence and multiplicity of classical solutions of a Riemannian problem.

**Proposition 4.1** *For any PFW the two following properties are equivalent:*

- (a) *geodesic connectedness (i.e., each two of its points can be joined by a geodesic);*
- (b) *the problem*

$$\begin{cases} D_s \dot{x}(s) = -\nabla_x V_\Delta(x(s), s) & \text{for all } s \in I \\ x(0) = x_0, \quad x(1) = x_1, \end{cases} \quad (4.1)$$

*admits a solution for all  $x_0, x_1 \in \mathcal{M}_0$ , all the possible values  $\Delta u \in \mathbb{R}$  and all the initial points  $u_0 = u(0)$ , where  $V_\Delta(x, s)$  is given in (3.2).*

**Proof.** (a)  $\Rightarrow$  (b) Recall that if a geodesic connects two given points, it can be reparametrized so to make its domain equal to  $I$ ; then, use Proposition 3.1.

(b)  $\Rightarrow$  (a) Fixed two points  $z_0 = (x_0, v_0, u_0)$ ,  $z_1 = (x_1, v_1, u_1)$ , the connecting geodesic  $z(s) = (x(s), v(s), u_0 + s\Delta u)$ ,  $\Delta u = u_1 - u_0$ , is obtained taking  $x(s)$  as solution of (4.1) and  $v(s)$  as in Proposition 3.1(c) with

$$E_z = 2(v_1 - v_0)\Delta u + 2 \int_0^1 \left( \frac{1}{2} \langle \dot{x}, \dot{x} \rangle - V_\Delta(x, s) \right) ds. \quad \blacksquare \quad (4.2)$$

Even though we are always considering differentiable curves, it is convenient now to introduce a wider space of curves which is useful for a variational approach to (4.1). Fixed two points  $x_0, x_1 \in \mathcal{M}_0$ , define the set

$$\Omega^1(x_0, x_1) = \{x \in H^1(I, \mathcal{M}_0) : x(0) = x_0, x(1) = x_1\},$$

where  $H^1(I, \mathcal{M}_0)$  is the Sobolev space containing the absolutely continuous curves from  $I$  to  $\mathcal{M}_0$  with finite integral of  $\langle \dot{x}, \dot{x} \rangle$ . It is well-known (see, for example, [10, Proposition 2.2]) that a curve  $x \in \Omega^1(x_0, x_1)$  is a classical solution of the problem (4.1) if and only if it is a critical point of the functional

$$\mathcal{J}_\Delta : x \in \Omega^1(x_0, x_1) \longmapsto \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle ds - \int_0^1 V_\Delta(x, s) ds \in \mathbb{R}. \quad (4.3)$$

Even if the existence of critical points for this functional is the most classical problem in calculus of variations, its complete solution for a natural case as the one we are interested ( $V_\Delta$  is differentiable and may behave quadratically at infinity) has been obtained only very recently by the authors by means of variational methods and Ljusternik–Schnirelmann theory (see [10]). In particular, [10, Theorem 1.1] implies:

**Proposition 4.2** *Let  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  be a complete (connected)  $n$ -dimensional Riemannian manifold. Let  $V \in C^1(\mathcal{M}_0 \times I, \mathbb{R})$  be such that*

$$V(x, s) \leq \lambda d^2(x, \bar{x}) + \mu d^p(x, \bar{x}) + k \quad \text{for all } (x, s) \in \mathcal{M}_0 \times I, \quad (4.4)$$

*for some  $p \in [0, 2[$ ,  $\bar{x} \in \mathcal{M}_0$  and (positive) real numbers  $\lambda, \mu, k$ .*

*If  $\lambda < \pi^2/2$ , then for all  $x_0, x_1 \in \mathcal{M}_0$  there exists at least one solution of the corresponding problem (4.1) which is an absolute minimum of  $\mathcal{J}_\Delta$ .*

*Moreover, if  $\mathcal{M}_0$  is not contractible in itself, there exists a sequence of solutions  $\{x_k\}_k$  such that  $\mathcal{J}_\Delta(x_k) \rightarrow +\infty$  if  $k \rightarrow +\infty$ .*

Hence, the following result in PFWs can be stated:

**Theorem 4.3** *Let  $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ ,  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ , be a PFW and fix  $u_0, u_1 \in \mathbb{R}^2$ , with  $u_0 \leq u_1$ . Suppose that:*

*(H<sub>1</sub>)  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$  is a complete  $n$ -dimensional Riemannian manifold;*

*(H<sub>2</sub>) there exist  $p \in [0, 2[$ ,  $\bar{x} \in \mathcal{M}_0$  and (positive) real numbers  $R_0, R_1, R_2$  such that for all  $(x, u) \in \mathcal{M}_0 \times [u_0, u_1]$  it is*

$$H(x, u) \geq -(R_0 d^2(x, \bar{x}) + R_1 d^p(x, \bar{x}) + R_2). \quad (4.5)$$

*Then, two points  $z_0 = (x_0, v_0, u_0), z_1 = (x_1, v_1, u_1) \in \mathcal{M}$  can be joined by a geodesic if*

$$R_0(u_1 - u_0)^2 < \pi^2. \quad (4.6)$$

*Moreover, under these hypotheses, if  $\mathcal{M}_0$  is not contractible in itself, then there exist infinitely many spacelike geodesics connecting  $z_0$  and  $z_1$ .*

**Proof.** It is straightforward from Propositions 4.1 and 4.2. To check that, in the case  $\mathcal{M}_0$  non-contractible, infinitely many *spacelike* geodesics  $\{z_k\}_k$  exist, recall that, using (4.3), the value of their energy  $E_{z_k} = E_z(x_k)$  in (4.2) is

$$E_z(x_k) = 2\Delta v \Delta u + 2\mathcal{J}_\Delta(x_k),$$

where  $\Delta v$  and  $\Delta u$  are constant and  $2\mathcal{J}_\Delta(x_k) \rightarrow +\infty$  if  $k \rightarrow +\infty$ . ■

**Remark 4.4** (1) Inequality (4.5) does not impose any condition on the growth of  $H(x, u)$  with respect to  $u$ , because we assume that  $u$  lies in the compact interval  $[u_0, u_1]$ . That is, one can consider  $R_0, R_1, R_2$  as continuous functions of  $u$  and impose (H<sub>2</sub>) for  $x \in \mathcal{M}_0$  at each fixed  $u \in [u_0, u_1]$ . Then, the bound (4.6) would be obtained just putting  $R_0 = \max\{R_0(u) : u \in [u_0, u_1]\}$ .

(2) Hypothesis (H<sub>2</sub>) can be deduced as a consequence of alternative inequalities involving either the  $\mathcal{M}_0$ -gradient  $\nabla_x$  or Hessian  $\text{Hess}_x$  of  $H$ . In fact, (4.5) will hold if one of the following two conditions is satisfied:

(i)  $H \in C^1(\mathcal{M}_0 \times [u_0, u_1], \mathbb{R})$  and for all  $(x, u) \in \mathcal{M}_0 \times [u_0, u_1]$  it is

$$\langle \nabla_x H(x, u), \nabla_x H(x, u) \rangle^{1/2} \leq 2R_0 d(x, \bar{x}) + R_1; \quad (4.7)$$

- (ii)  $H \in C^2(\mathcal{M}_0 \times [u_0, u_1], \mathbb{R})$  and there exists  $K \geq 0$  such that for all  $x \in \mathcal{M}_0$  with  $d(x, \bar{x}) \geq K$  it is

$$\text{Hess}_x H(x, u)[\xi, \xi] \leq 2R_0 \langle \xi, \xi \rangle \quad \text{for all } \xi \in T_x \mathcal{M}_0, u \in [u_0, u_1] \quad (4.8)$$

(it is also possible to replace each one of these inequalities by a lim sup, as in [10, Remark 1.2]).

Remarkably, (4.7) as well as (4.8) imply also the different inequality for  $H$  which was needed for the result on completeness stated in Corollary 3.4.

Notice that, if  $(H_2)$  holds in each compact interval  $[u_0, u_1]$  with  $R_0 = 0$ , then the geodesic connectedness is obtained. More precisely:

**Corollary 4.5** *A PFW is geodesically connected if  $(H_1)$  in Theorem 4.3 holds and*

- $(H'_2)$  there exist  $\bar{x} \in \mathcal{M}_0$ , (positive) continuous functions  $R_1(u), R_2(u)$  and  $p(u) < 2$  such that for all  $(x, u) \in \mathcal{M}_0 \times \mathbb{R}$  it is*

$$H(x, u) \geq -(R_1(u)d^{p(u)}(x, \bar{x}) + R_2(u)). \quad (4.9)$$

The following example (inspired by [10, Example 3.6]) proves that condition (4.6) is the best estimate one can obtain if  $H$  grows quadratically with respect to  $x$ .

**Example 4.6** Consider an (exact) PFW,  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ ,  $\mathcal{M}_0 = \mathbb{R}^n$  with  $H(x, u) = -|x|^2$ . Obviously, condition  $(H_2)$  is satisfied ( $R_0 = 1$ ,  $R_1 = R_2 = 0$ ,  $\bar{x} = 0$ ) and Theorem 4.3 ensures geodesic connectedness for any pair of points  $z_0 = (x_0, v_0, u_0)$ ,  $z_1 = (x_1, v_1, u_1)$  such that  $|u_1 - u_0| < \pi$ . On the contrary, there are non-geodesically connectable points  $z_0, z_1$  with  $|\Delta u| = |u_1 - u_0| = \pi$ . In fact, recall that in this model it is  $V_\Delta(x(s)) = \frac{\pi^2}{2}|x(s)|^2$  and the corresponding Riemannian problem (4.1) becomes

$$\begin{cases} \ddot{x}(s) + \pi^2 x(s) = 0 \\ x(0) = x_0, \quad x(1) = x_1. \end{cases}$$

Clearly, taken  $x_0 = 0$  and  $x_1 \neq 0$ , this problem has no solution and the corresponding two points in  $\mathcal{M}$  are non-connectable.

## 4.2 Connectedness by causal geodesics

Now, in order to give an Avez-Seifert type result, let us recall that any PFW  $\mathcal{M}$  is time oriented (see Section 2) and, fixed any  $z_0 = (x_0, v_0, u_0) \in \mathcal{M}$ , its causal future is defined as

$$J^+(z_0) = \{z \in \mathcal{M} : z = z_0 \text{ or there is a future directed piecewise smooth causal curve in } \mathcal{M} \text{ from } z_0 \text{ to } z\}.$$

Thus, if  $z_1 = (x_1, v_1, u_1) \in J^+(z_0)$ , then  $u_0 \leq u_1$  by (2.1).

**Theorem 4.7** *Let  $(\mathcal{M}, \langle \cdot, \cdot \rangle)$  be a PFW,  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ . Fix  $z_0 = (x_0, v_0, u_0)$ ,  $z_1 = (x_1, v_1, u_1) \in \mathcal{M}$  with  $z_1 \in J^+(z_0)$ .*

*Assume that  $(H_1)$ ,  $(H_2)$  and (4.6) in Theorem 4.3 hold.*

*Then, there exists a future directed causal geodesic from  $z_0$  to  $z_1$  which has maximum length among all the causal curves connecting these endpoints.*

**Proof.** Let  $z : I \rightarrow \mathcal{M}$ ,  $z(s) = (x(s), v(s), u(s))$ , be a future directed causal curve such that  $z(0) = z_0$ ,  $z(1) = z_1$ . We can assume  $u_0 < u_1$ . Otherwise, as  $\dot{u}(s) \geq 0$  (see (2.1)),  $u(s)$  would be constant and  $z(s)$  must be a lightlike pregeodesic of  $\Pi_{u_0}$  (see Corollary 3.6). We will also assume  $\dot{u}(s) > 0$  for all  $s$  because, otherwise, from standard arguments in Causality Theory a longer timelike curve with the same endpoints could be found. Moreover,  $z(s)$  will be considered reparametrized in such a way that  $\dot{u}(s)$  is constant, i.e.,  $s = (u(s) - u_0)/\Delta u$ ,  $\Delta u = u_1 - u_0$ .

By Proposition 4.2 and Theorem 4.3,  $z_0$  and  $z_1$  can be joined by a geodesic  $\bar{z} : I \rightarrow \mathcal{M}$ ,  $\bar{z}(s) = (\bar{x}(s), \bar{v}(s), \bar{u}(s))$ , such that  $\bar{x}$  is a minimum point of  $\mathcal{J}_\Delta$  in  $\Omega^1(x_0, x_1)$ , while  $\bar{u}$  and  $\bar{v}$  are defined as in Proposition 3.1 (a), (c). Our aim is to prove that

$$E_{\bar{z}} \leq \int_0^1 E_z(s) ds, \quad (4.10)$$

where  $E_{\bar{z}}$  is the energy of the geodesic  $\bar{z}$ , as computed from (4.2), and  $E_z(s) = \langle \dot{z}(s), \dot{z}(s) \rangle (\leq 0)$  on  $I$ . In fact, inequality (4.10) proves not only that  $\bar{z}(s)$  is causal but also that it has minimum energy among the considered curves and, by a standard application of Cauchy-Schwarz inequality, maximum length among all connecting causal curves.

Now, recall that  $\mathcal{J}_\Delta(\bar{x}) \leq \mathcal{J}_\Delta(x)$  and, using (3.2), (3.6), (4.2) and (4.3):

$$\begin{aligned} E_{\bar{z}} &\leq 2(v_1 - v_0)\Delta u + 2\mathcal{J}_\Delta(x) \\ &= 2 \int_0^1 \left( \dot{v}\dot{u} + \frac{1}{2} \langle \dot{x}, \dot{x} \rangle - V_\Delta(x, s) \right) ds = \int_0^1 E_z(s) ds, \end{aligned}$$

as required. ■

As in the case of geodesic connectedness, our hypotheses for connection under causal curves are sharp, as the following example shows.

**Example 4.8** Let  $(\mathbb{R}^{n+2}, \langle \cdot, \cdot \rangle)$  be the PFW introduced in Example 4.6. We have already remarked that  $(H_1), (H_2)$  are satisfied but, for example, the points  $z_0 = (0, 0, 0)$  and  $z_1 = (x_1, v_1, \pi)$ ,  $x_1 \neq 0$  (which do not satisfy (4.6)) cannot be connected by a geodesic for any value of  $v_1$ . Anyway,  $v_1$  can be chosen such that  $z_1 \in J^+(z_0)$ . In fact, taken  $v_1 < 0$  with  $|v_1|$  large enough, the connecting curve  $z(s) = s \cdot (x_1, v_1, \pi)$ ,  $s \in I$ , is causal.

**Remark 4.9** All the remarks to the hypotheses of Theorem 4.3 still hold. In particular, in the case that, instead of  $(H_2)$ , the stronger assumption  $(H'_2)$  in Corollary 4.5 holds, then inequality (4.6) will hold automatically; thus, two points will be causally related if and only if they can be joined by a causal (length-maximizing) geodesic. In the forthcoming article [19] hypotheses  $(H_1)$

and  $(H'_2)$  are shown to imply global hyperbolicity, and, thus, in this case the result on existence of causal geodesics can be obtained as a consequence of classical Avez–Seifert’s one. Nevertheless, our proof of Theorem 4.7 is based on completely different arguments, and we obtain not only the result in the non–globally hyperbolic case  $(H_2)$  but also some multiplicity results on timelike geodesics, as we show next (see Theorem 4.10 and Corollary 4.11). On the other hand, the proof of global hyperbolicity can be simplified by using Theorem 4.7 (see [19, Theorem 4.1, Remark 4.4]).

If the Riemannian part of a PFW is topologically non–trivial, it has been proved the existence of infinitely many spacelike geodesics (see Theorem 4.3). Now, under the same assumptions, we are able to prove some multiplicity results for timelike geodesics.

**Theorem 4.10** *Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  be a PFW. Fix  $z_0 = (x_0, v_0, u_0) \in \mathcal{M}$  and  $(x_1, u_1) \in \mathcal{M}_0 \times \mathbb{R}$  such that  $\Delta u = u_1 - u_0 \neq 0$ .*

*Assume that  $(H_1)$ ,  $(H_2)$  and (4.6) in Theorem 4.3 hold and  $\mathcal{M}_0$  is not contractible in itself.*

*Then, either*

$$\begin{aligned} \lim_{v \rightarrow -\infty} N(z_0, z_v) &= +\infty & \text{if } \Delta u > 0 & \quad \text{or} \\ \lim_{v \rightarrow +\infty} N(z_0, z_v) &= +\infty & \text{if } \Delta u < 0, \end{aligned}$$

where  $z_v = (x_1, v, u_1)$ ,  $v \in \mathbb{R}$ , and  $N(z_0, z_v)$  is the number of timelike geodesics from  $z_0$  to  $z_v$ .

**Proof.** In order to prove the multiplicity result, let us recall some more details about the variational and topological tools which are needed in the proof of Proposition 4.2. In fact, if  $\mathcal{M}_0$  is not contractible in itself, by a Fadell and Husseini’s result (cf. [18]) it follows that the manifold of curves  $\Omega^1(x_0, x_1)$  has infinite Ljusternik–Schnirelman category and contains compact subsets of arbitrarily high category. Whence, since the classical Ljusternik–Schnirelmann Theorem applies to  $\mathcal{J}_\Delta$  in the given assumptions (see [10]), such a functional has infinitely many critical points  $\{x_k\}_k$  such that

$$\mathcal{J}_\Delta(x_k) = \inf_{A \in \Gamma_k} \sup_{x \in A} \mathcal{J}_\Delta(x) \quad \text{for each } k \in \mathbb{N}, k \geq 1, \quad (4.11)$$

with

$$\Gamma_k = \{A \subseteq \Omega^1(x_0, x_1) : \text{cat}_{\Omega^1(x_0, x_1)}(A) \geq k\}$$

(here,  $\text{cat}_{\Omega^1(x_0, x_1)}(A)$  is the Ljusternik–Schnirelmann category of  $A$  with respect to  $\Omega^1(x_0, x_1)$ , i.e., the least number of closed and contractible subsets of  $\Omega^1(x_0, x_1)$  covering  $A$ ; for more details see, e.g., [28] or also [25, Section 2.6]). Obviously, by the definition (4.11) it follows that

$$\mathcal{J}_\Delta(x_1) \leq \mathcal{J}_\Delta(x_2) \leq \dots \leq \mathcal{J}_\Delta(x_k) \leq \dots$$

On the other hand, fixed  $t \in \mathbb{R}$  and  $z_t = (x_1, t, u_1)$ , Proposition 3.1 and (4.2), (4.3) imply that for any  $k \geq 1$  the curve  $z_t^k(s) = (x_k(s), v_t^k(s), u_0 + s\Delta u)$ ,  $s \in I$ , is a geodesic in  $\mathcal{M}$  with energy

$$E_t^k = 2(t - v_0)\Delta u + 2J_\Delta(x_k) \quad (4.12)$$

if  $v_t^k(s)$  is as in Proposition 3.1(c).

Now, let  $m \in \mathbb{N}$ ,  $m \geq 1$ , be fixed. Whence, by (4.12), if  $\Delta u > 0$  there exists  $t_m < v_0$  such that for all  $t \leq t_m$  the corresponding  $z_t^1, z_t^2, \dots, z_t^m$  are  $m$  timelike geodesics joining  $z_0$  to  $z_t$ ; while, on the other hand, if  $\Delta u < 0$  the same result holds but choosing  $t_m > v_0$  and  $t \geq t_m$ . ■

If we replace condition  $(H_2)$  in Theorem 4.10 by the stronger one  $(H'_2)$ , then the further assumption (4.6) holds automatically since it is  $R_0 = 0$  (in this case, the functional  $\mathcal{J}_\Delta$  is bounded from below for every  $\Delta u = u_1 - u_0$ ); thus, not only the same results of Theorem 4.10 still hold but the same arguments of its proof allow to prove a second multiplicity estimate. More precisely, it can be proved that:

**Corollary 4.11** *Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  be a PFW. Fix  $z_0 = (x_0, v_0, u_0) \in \mathcal{M}$  and  $(x_1, v_1) \in \mathcal{M}_0 \times \mathbb{R}$ .*

*Assume that  $(H_1)$  and  $(H'_2)$  in Corollary 4.5 hold and let  $\mathcal{M}_0$  be not contractible in itself.*

*Then, either*

$$\begin{aligned} \lim_{u \rightarrow -\infty} N(z_0, z_u) &= +\infty & \text{if } v_1 > v_0 & \quad \text{or} \\ \lim_{u \rightarrow +\infty} N(z_0, z_u) &= +\infty & \text{if } v_1 < v_0, \end{aligned}$$

where  $z_u = (x_1, v_1, u)$ ,  $u \in \mathbb{R}$ , and  $N(z_0, z_u)$  is the number of timelike geodesics from  $z_0$  to  $z_u$ .

### 4.3 Application to exact gravitational waves

At last, we want to apply the previous results to the classical models of exact gravitational waves. To this aim, the previous Theorems 4.3, 4.7, 4.10, Corollary 4.11 and Remarks 4.4(1), 4.9 can be summarized as follows:

**Corollary 4.12** *Let  $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ ,  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ , be a PFW such that  $(H_1)$  holds and fix  $\bar{x} \in \mathcal{M}_0$ . Let  $R_0(u), R_1(u), R_2(u) (\geq 0)$ ,  $p(u) < 2$  be continuous functions and put, for any  $u_0, u_1 \in \mathbb{R}$  (not necessarily  $u_0 \leq u_1$ ):*

$$R_0[u_0, u_1] = \text{Max}\{R_0(u) : u \in [u_0, u_1] \cup [u_1, u_0]\}$$

*(with  $[u_i, u_j] = \emptyset$  if  $u_i > u_j$ ,  $i, j \in \{0, 1\}$ ). Then,*

- (1) *if  $H(x, u) \geq -(R_1(u)d^{p(u)}(x, \bar{x}) + R_2(u))$  for all  $(x, u) \in \mathcal{M}_0 \times \mathbb{R}$ , then  $\mathcal{M}$  is geodesically connected;*

- (2) if  $H(x, u) \geq -(R_0(u)d^2(x, \bar{x}) + R_1(u)d^{p(u)}(x, \bar{x}) + R_2(u))$  for all  $(x, u) \in \mathcal{M}_0 \times \mathbb{R}$ , two points  $z_0 = (x_0, v_0, u_0)$ ,  $z_1 = (x_1, v_1, u_1) \in \mathcal{M}$  can be surely connected by means of a geodesic if

$$R_0[u_0, u_1](u_1 - u_0)^2 < \pi^2. \quad (4.13)$$

Moreover, when either the case (1) or the case (2) holds:

- (a) if  $z_1 \in J^+(z_0)$  there exists a length-maximizing causal geodesic connecting  $z_0$  and  $z_1$ ;
- (b) if  $\mathcal{M}_0$  is not contractible in itself:
  - (i) there exist infinitely many spacelike geodesics connecting  $z_0$  and  $z_1$ ,
  - (ii) the number of timelike geodesics from  $z_0$  to  $z_v = (x_1, v, u_1)$  goes to infinity when  $v \rightarrow -\infty$  if it is  $u_1 > u_0$  or when  $v \rightarrow +\infty$  if it is  $u_1 < u_0$ .

Furthermore, only in the case (1), if  $\mathcal{M}_0$  is not contractible in itself the number of timelike geodesics from  $z_0$  to  $z_u = (x_1, v_1, u)$  goes to infinity when  $u \rightarrow -\infty$  if it is  $v_1 > v_0$  or when  $u \rightarrow +\infty$  if it is  $v_1 < v_0$ .

In particular, Corollary 4.12 is applicable to exact gravitational waves as follows:

**Proposition 4.13** *Let  $(\mathbb{R}^4, ds^2)$  be an exact gravitational plane wave. Then, the case (2) of Corollary 4.12 holds with*

$$R_0[u_0, u_1] = \text{Max}\{(f^2 + g^2)^{1/2}(u) : u \in [u_0, u_1] \cup [u_1, u_0]\} \quad \text{and } R_1, R_2 \equiv 0.$$

**Proof.** Recall that, for an exact gravitational wave,  $H(\cdot, u)$  is a quadratic form with eigenvalues  $\pm(f^2 + g^2)^{1/2}(u)$ . Thus, for all  $x = (x_1, x_2)$ , we have  $H(x, u) \geq -(f^2 + g^2)^{1/2}(u)|x|^2$  (the equality holds for the corresponding eigenvectors). ■

**Remark 4.14** Fixed  $u_0$ , the function  $R_0[u_1] := R_0[u_0, u_1]$  cannot decrease when  $|u_1 - u_0|$  grows. Thus, on any exact gravitational wave the left hand side of (4.13) must reach the value  $\pi^2$  for some values of  $u_1$ , i.e., we can find unique  $u_1^-, u_1^+ \in \mathbb{R}$  such that:

- (i)  $u_1^- < u_0 < u_1^+$  and
- (ii)  $R_0[u_1^-] = \frac{\pi^2}{(u_1^- - u_0)^2}$ ,  $R_0[u_1^+] = \frac{\pi^2}{(u_1^+ - u_0)^2}$ .

Thus, Corollary 4.12 is applicable whenever  $u \in ]u_1^-, u_1^+[$ . In particular, this yields a bound for the appearance of the first astigmatic conjugate point (see [4, pp. 486]).

Even more, using the same idea of previous examples, we can check that our hypotheses are the sharpest ones, even for exact polarized sandwich waves.

**Example 4.15** Let  $(\mathbb{R}^4, ds^2)$  be an exact gravitational wave such that in (1.3) it is  $f(u) = 1$  on  $[0, \pi]$ ,  $f(u) = 0$  out of a compact subset and  $g(u) \equiv 0$ . Choose the points  $z_0 = (0, 0, 0, 0)$  and  $z_1 = (x_1, 0, v_1, \pi)$  with  $x_1 \neq 0$ . The same arguments in Examples 4.6, 4.8 show that  $z_0$  and  $z_1$  cannot be connected by a geodesic for any  $v_1 \in \mathbb{R}$ , even in the case that  $-v_1 > 0$  is large enough such to imply  $z_1 \in J^+(z_0)$ .



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